Systemic Risk Contributions: A Credit Portfolio Approach☆

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Abstract

We put forward a Merton-type multi-factor portfolio model for assessing banks' contributions to systemic risk. This model accounts for the major drivers of banks' systemic relevance: size, default risk and correlation of banks' assets as a proxy for interconnectedness. We measure systemic risk in terms of the portfolio expected shortfall (ES). Banks' (marginal) risk contributions are calculated based on partial derivatives of the ES in order to ensure a full risk allocation among institutions. We compare the performance of an importance sampling algorithm with a fast analytical approximation of the ES and the marginal risk contributions. Furthermore, we show empirically for a portfolio of large international banks how our approach could be implemented to compute bank-specific capital surcharges for systemic risk or stabilisation fees. We find that size alone is not a reliable proxy for the systemic importance of a bank in this framework. In order to smooth cyclical fluctuations of the risk measure, we explore a time-varying confidence level of the ES.

Keywords: systemic risk contribution, systemic capital charge, expected shortfall, importance sampling, granularity adjustment

JEL: C15, C63, E58, G01, G21

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1. Introduction

The failure of certain financial institutions such as Lehman, Northern Rock or HRE during the crisis of 2007-2009 highlighted the significant adverse impact that a failure of a single firm can have on the financial system as a whole. Therefore, a firm-specific or microprudential approach is not sufficient to promote financial stability. Instead a careful assessment of a

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financial firm's *contribution* to the system-wide risk should be an important part of macro-prudential financial supervision.

The risk that refers to a financial system as a whole is often addressed as *systemic risk*. We define this term in the following as *the risk of a collapse of a financial system that entails a social welfare loss*. The task of addressing a systemic event and its negative externalities requires approaches for *measuring* system-wide risk and decomposing it into the *contributions* of individual institutions. A macro-prudential approach would rely on measures of the magnitude of the potential loss or cost associated with the systemic event and on procedures for building up a sufficient capital basis in the financial system to bear (most of) this cost. As an important auxiliary condition, macro-prudential measures should contribute to reduce potential procyclical effects of regulation.

In this paper we focus on the subject of measuring and allocating systemic risk. For this purpose we propose a widely used credit risk model that treats the financial system of banks similar to a portfolio of securities and takes into account interlinkages between banks through their asset correlations. Furthermore, the multi-factor correlation structure allows for a differentiated treatment of individual or certain groups of institutions. This reflects the fact that episodes of financial distress often arise from the exposure of groups of institutions to common risk factors.

In the portfolio context, a systemic event corresponds with the realisation of extreme portfolio losses. The maximum systemic risk tolerated is defined as the *expected shortfall* (ES) at a confidence level q, i.e., the expected loss in the worst $100(1-q)\%$ of cases. The value of the confidence level q is set by the regulator depending on his risk tolerance. A macro-prudential tool based on the ES may generate procyclical effects because of cyclical risk components such as point-in-time default probabilities. Therefore, we consider also a time-variant confidence level $q(t)$ as a possible mitigant of procyclicality.

In order to break down extreme portfolio losses into the contributions of individual banks we draw on a rich literature on coherent, additive risk contributions for credit portfolios. Employing marginal risk contributions based on the partial derivatives of the portfolio ES with respect to the institutions' relative portfolio weight allows for a complete allocation of the system-wide risk to the individual banks.

On the basis of the estimated system-wide tail risk and its decomposition into the individual institutions' contributions, a set of rule-based policy interventions, such as systemic capital charge or a stabilisation fee, can be designed.

In summary, we see the following four aspects as the main contribution of this paper:

1. We provide a full allocation of the systemic risk across institutions based on the Euler allocation principle thereby adopting a methodology that is well-researched in the risk management literature for the assessment of banks' systemic importance.
2. We derive an analytical approximation of the marginal risk contributions and compare its performance with a simulation-based importance sampling technique.
3. We use equity market information in order to gauge the market participants' collective evaluation of the otherwise difficult to quantify interlinkages that drive systemic risk.
4. We propose and empirically explore a time-varying confidence level of the ES as a method to mitigate procyclical effects of capital charges for systemic risk.

The remainder of the paper is organised as follows. Section 2 provides a brief review of selected literature. Section 3 presents the credit portfolio model on which the tail risk contributions are based. Sections 4 and 5 present the estimation of the system-wide tail risk as well as tail risk contributions by means of an IS simulation and an analytical solution respectively. Section 6 reports the results of an empirical study carried out for a sample of large international banks. In section 7 the risk drivers of the systemic risk and the banks' respective systemic importance are analysed, namely the probability of default, the asset correlations, and the relative size of a bank in the financial system. Possible policy implications of the proposed methodology are presented in section 8. In this section we distinguish between two dimensions: a cross-sectional dimension including a proposal for an ES-based capital surcharge for systemic risk and a time series dimension in which we smooth the cyclicality of the risk measure by a time-variant confidence level. Section 9 summarises and concludes.

2. Related literature

Many methods for assessing systemic risk and risk contributions have been discussed in the related literature. The IMF's Global Financial Stability Report (IMF, 2009, pp 73-149) reviews the most recent approaches for detecting the tail risk of a financial system by examining direct and indirect financial sector interlinkages. Market prices of financial instruments and credit risk modelling have already been used in the literature in order to measure systemic risk.

De Nicolo and Kwast (2002) argue that the information contained in banks' equity returns can be used to measure the total (direct and indirect) dependence since stock prices reflect market participants' collective evaluation of the future prospects of the firm, including the total impact of its interactions with other institutions. In our paper we incorporate the banks' equity return correlation in order to judge the correlation between the institutions' defaults.

Equity returns and other market data are widely used to measure the fragility of financial institutions at individual and aggregate levels. For example, Bartram et al. (2007) estimate the default probabilities for a large sample of international banks from time series of equity prices and also from equity option prices, based on the assumptions of Merton's structural model (Merton, 1974). They use this information to construct indicators for a systemic event. In our paper we use the estimates of banks' default probabilities obtained from Moody's KMV, whose model is also based on the Merton's fundamental idea.

Huang et al. (2009) deduce risk neutral default probabilities for major banks from their CDS spreads and asset return correlation from the co-movement of equity returns. Using these key parameters as input in a portfolio credit risk model, the authors suggest computing an indicator of systemic risk, namely the price of insurance against large default losses in the banking sector. The theoretical insurance premium equals the risk-neutral expectation of portfolio credit losses given that the losses exceed some minimum share of the sector's total liabilities.

Another application of the credit portfolio approach based on market data can be found in Segoviano and Goodhart (2009). The authors utilise the "nonparametric consistent information multivariate density optimising methodology" in order to obtain the joint multivariate density of the banks' asset value movements. Based on this information, several indicators of banking stability can be constructed: (i) the joint probability of distress of all banks in the portfolio; (ii) a banking stability index that reflects the expected number of banks becoming distressed given at least one bank has become distressed; (iii) the conditional probabilities of distress for individual banks or specific groups of banks.

Also by virtue of the joint probability distribution of banks' assets, Lehar (2005) specifies the following indicators of systemic risk: (i) an asset-value-related systemic risk index by computing the probability that a group of banks with a total amount of assets greater than a certain fraction of all banks' assets goes bankrupt within a short period of time; (ii) a number-of-defaults-related systemic risk index by computing the probability that a certain number of banks go bankrupt within a short period of time; (iii) the value of a hypothetical deposit insurance, its volatility as well as the individual volatility contributions.

While most of the methods described above focus on the monitoring of systemic risk, Adrian and Brunnermeier (2009) suggest an approach for measuring the *contributions* that individual banks make to systemic risk. For this purpose the authors make use of the quantile regression technique and the CoVaR measure. The authors suggest predicting individual risk contributions on the basis of certain firm-specific characteristics like size, leverage and maturity mismatch. A shortcoming of the CoVaR approach is that the sum of individual risk contributions does not equal the system-wide risk.

Tarashev et al. (2010) use a game theoretic concept and allocate systemic risk contributions to banks based on the Shapley value concept. Their methodology can in principle be applied either with VaR or ES as the relevant risk measure of the financial system. Banks' individual probabilities of default, (a fixed fraction of) the book value of liabilities and the chosen asset correlation coefficient entirely determine the probability distribution of portfolio losses and allow for the estimation of the portfolio tail risk. The risk at the portfolio level is then attributed to the individual institutions by means of the Shapley value methodology. Thereby, for each specific institution, its contribution to the risk of all possible subportfolios in which this institution is present have to be computed. The average value of all those contributions is than the institution's contribution to the systemic risk, or its Shapley value. The authors suggest to use the Shapley value as a measure of an institution's systemic importance on whose basis macro-prudential policy interventions may be conducted. Unfortunately, due to the rapidly increasing computational complexity, the Shapley value methodology can only be applied to very small portfolios or portfolios consisting of few homogeneous subportfolios. The approach put forward in this paper instead remains feasible for large and heterogenous portfolios (or financial systems). Furthermore, compared to the one-factor asset return decompositon adopted by Tarashev et al. (2010) the utilisation of a multi-factor model allows for a more risk-sensitive modelling of systemic risk.

Another proposal how to measure financial institutions' contribution to systemic risk is put forward by Acharya et al. (2010). Their *marginal expected shortfall* measure is conceptually related to our approach but defined differently: In order to facilitate its computation they define this measure by the worst 5% net equity returns at daily frequency. In this paper we use the marginal expected shortfall in the sense of the Euler allocation principle, based on the portfolio risk characteristics at a certain point in time, instead of a time-series estimate based on past equity returns. Acharya et al. (2010) also embed their risk measure into an economic model to determine an optimal taxation policy for systemic risk which is an extension not addressed in our paper.

Since the focus of this paper is on the application of the credit portfolio methodology using market and balance sheet data, we refer to the IMF's GFSR (IMF, 2009) as well as the references therein for more research on network analysis and domino effects. Moreover, De Bandt and Hartmann (2000) provide a comprehensive survey on the theoretical and empirical literature on contagion in banking and financial markets as well as in payment and settlement systems. See also Nier et al. (2007) for further useful references. An example of an integrated systemic risk framework which combines standard techniques from market and credit risk management with a network model of a banking system is the OeNB's Systemic Risk Monitor, see Boss et al. (2006).

3. Model set-up

We think of a financial system as a portfolio of n assets, the assets being financial institutions. The portfolio's loss distribution describes the risk of the entire financial system. Losses can only be induced by a distress of one or more institutions included in the portfolio. For the $i-$th institution, the exposure at distress, EAD_i, is defined as the book value of the institution's *liabilities* that are defined in this paper in nominal terms and after deducting capital. Then $w_i = EAD_i/\sum_{i=1}^n EAD_i$ denotes the relative portfolio weight of the ith institution. The loss given distress, LGD_i, represents a fraction of the total liabilities which specifies the potential costs of the resolution or a bail-out of the distressed financial institution. An event of distress occurs at a predefined time horizon with the unconditional *distress probability* p_i. The event of distress is captured by the Bernoulli random variable $D_i \sim Be(p_i)$. In the spirit of the structural credit risk framework, we define distress as an event when the asset return of a financial institution hits or falls below its default threshold at a pre-specified time horizon. The default threshold specifies the point where the institution has to either enter resolution or be bailed out.

To complete the asset value model, we further assume that the standardised asset returns $\{X_i\}_{i=1,\ldots,n}$ are multivariate normally distributed with a full-rank correlation matrix. To explain where the linear dependence results from, we decompose $\{X_i\}$ into a systematic and an idiosyncratic component by means of a multi-factor model. Following Pykhtin (2004) we assume that the asset return of a financial institution i depends on a composite systematic risk factor Y_i which is a convex combination of a set of independent standard normally distributed systematic risk factors $\{Z_k\}_{k=1,\ldots,m}$ with $m \ll n$. The idiosyncratic part of the asset return variation is captured by an independent standard normally distributed shock ϵ_i.

The model framework for the risk drivers $\{X_i\}_{i=1,\ldots,n}$, the distress indicators $\{D_i\}_{i=1,\ldots,n}$ and our target variable – the portfolio loss rate PL – can now be formally summarised as follows:

$$X_i = a_i Y_i + \sqrt{1-a_i^2}\,\epsilon_i, \qquad a_i \in (0,1) \tag{3.1}$$

$$Y_i = \sum_{k=1}^{m} \alpha_{ik} Z_k, \qquad \sum_{k=1}^{m} \alpha_{ik}^2 = 1 \tag{3.2}$$

$$Z_k, \epsilon_i \overset{iid}{\sim} N(0,1) \text{ for all } k = 1, \ldots, m \text{ and } i = 1, \ldots, n$$

$$D_i = 1 \Leftrightarrow X_i \in \big(-\infty, \Phi^{-1}(p_i)\big] \tag{3.3}$$

$$PL = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i. \tag{3.4}$$

In the expressions above, the factor loading a_i specifies the sensitivity of the particular institution to the systematic risk factor, and the asset correlation between distinct institutions i and j is given by $\rho_{i,j} = a_i a_j \rho_{Y_i,Y_j}$, where $\rho_{Y_i,Y_j} = \sum_{k=1}^{m} \alpha_{ik}\alpha_{jk}$ denotes the correlation between the two composite factors.

As already mentioned in section 1, we are primarily interested in the ES at a confidence level q as a coherent measure of the portfolio tail risk. But for the sake of completeness, we also report the results on VaR which defines the threshold for the ES measure. Let us denote the (discrete) cumulative distribution function of the portfolio loss rate by $F_{PL}(\cdot)$ and its quantile function by $F_{PL}^{-1}(\cdot)$. Then, VaR and ES can be defined as follows:

$$VaR_q(PL) = F_{PL}^{-1}(q) = \inf\big\{x \in [0,1] \,:\, F_{PL}(x) \geqslant q\big\} \tag{3.5}$$

$$ES_q(PL) = \frac{1}{1-q}\int_q^1 VaR_t(PL)dt. \tag{3.6}$$

As an alternative to (3.6), Kalkbrener (2005, p 434) considers an expression which turns out to be more instructive especially for simulation purposes later in this paper. As shown in Acerbi and Tasche (2002), if the distribution of portfolio loss were continuous, (3.6) would coincide with the tail conditional expectation (TCE) defined as

$$TCE_q(PL) = E\big[PL \mid PL \geqslant VaR_q(PL)\big]. \tag{3.7}$$

For a discrete loss distribution, however, the expression above has to be augmented with a correction term which adjusts the TCE measure upwards if the probability of the portfolio losses at the point $VaR_q(PL)$ does not coincide with q:

$$\begin{aligned} ES_q(PL) &= E\big[PL \mid PL \geqslant VaR_q(PL)\big] \\ &\quad + \frac{1}{1-q} VaR_q(PL)\big[F_{PL}\big(VaR_q(PL)\big) - q\big]. \end{aligned} \tag{3.8}$$

After computing the overall tail risk, we turn to the calculation of individual risk contributions which satisfy the full allocation property, i.e., their sum equals the total system-wide risk. For this purpose we use the *Euler allocation* or the marginal risk contributions based on the derivatives of the tail risk measure with respect to the portfolio weights of individual positions.

The marginal contributions measure the impact of a small change in the portfolio weight of a bank on the total tail risk of the whole portfolio. The Euler allocation principle has proved useful in portfolio-oriented risk management, particulary for the purpose of economic capital allocation, performance measurement, portfolio optimisation or risk-sensitive pricing. According to Denault (2001) the Euler allocation can also be motivated by game theory as the partial derivatives correspond to the Aumann-Shapley value that lies in the core of a coalitional game. For more information on the concept of Euler contributions as well as related literature and economic motivation see Tasche (2008). For an axiomatic approach to coherent risk measures and capital allocation see also Kalkbrener (2005).

In the following two sections we consider two methods to compute the portfolio risk and banks' risk contributions: firstly by simulation and afterwards by an analytical approximation.

4. Measuring and allocating systemic risk by simulation

Although it appears straightforward to compute an estimate of the portfolio tail risk by simulation, the brute-force Monte Carlo (MC) technique may fail for such rare events as $PL \geqslant VaR_q(PL)$. To clarify this point, let us consider the issue of estimating the small probability $1-q$ of a rare event which is, concerning simulation efficiency, equivalent to estimating VaR_q. That probability can be represented in terms of expectation, so that the MC estimator of $1-q$ would be the sample mean. The MC estimator is unbiased and normally distributed with variance $q(1-q)/s$, where s denotes the number of simulation runs. This means, for instance, that for the estimation of $1-q = 0.001$ with at most 5% relative error at the 95% confidence level, more than 1.5×10^6 simulation runs are necessary. Furthermore, individual losses conditional on the rare event $PL = VaR_q(PL)$ would be much more difficult to assess. Against this background the estimation of VaR contributions by MC would involve either unacceptable runtimes or high estimation errors. As has already been pointed out by Merino and Nyfeler (2004) and Glasserman (2006) among others, a similar problem arises when estimating ES contributions.

In order to reduce estimation errors, a plain MC simulation has to be modified, increasing the frequency of rare events while ensuring the estimator remains unbiased. A promising technique for the simulation of rare events and, therefore, for the estimation of the tail risk as well as the risk contributions is importance sampling (IS). For the Gaussian conditional independence framework, Glasserman and Li (2005) have already developed an appropriate two-stage IS algorithm leading to an asymptotically efficient estimator for a small probability $1-q$. Moreover, Glasserman (2006) provides further results on the IS estimation of VaR, ES and corresponding tail risk contributions. In AppendixA we provide an IS simulation algorithm for the portfolio loss distribution $F_{PL}(\cdot)$ within the Gaussian framework. On the basis of the simulated distribution $\hat{F}_{PL}(\cdot)$, tail risk measures and corresponding risk contributions can be estimated, as described in the following.

In order to estimate the VaR at a confidence level q, as defined in (3.5), the following expression can be used:

$$\widehat{VaR}_q(PL) = \inf\big\{x \in [0,1] \ : \ \hat{F}_{PL}(x) \geqslant q\big\}. \tag{4.1}$$

For the ES, according to (3.8), we obtain the estimator:

$$\widehat{ES}_q(PL) = \frac{\sum_{k=1}^{s} PL^k \, \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)}{\sum_{k=1}^{s} \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)} + \frac{1}{1-q}\widehat{VaR}_q(PL)\big[\hat{F}_{PL}\big(\widehat{VaR}_q(PL)\big) - q\big], \tag{4.2}$$

where k denotes one of s simulation runs.

Regarding a suitable IS estimator for the tail risk contributions, we refer to Tasche (2000) for the results on the additive contributions associated with quantile-based risk measures. The author proves that under certain continuity conditions imposed on the joint probability distribution of the individual loss variables $L_i = w_i \cdot LGD_i \cdot D_i$, the marginal contributions derived via differentiation of VaR and TCE can be represented in terms of the conditional expectation:

$$w_i \frac{\partial}{\partial w_i} VaR_q(PL) = E\big[L_i \mid PL = VaR_q(PL)\big] \tag{4.3}$$

and

$$w_i \frac{\partial}{\partial w_i} TCE_q(PL) = E\big[L_i \mid PL \geqslant VaR_q(PL)\big]. \tag{4.4}$$

Obviously, the risk contributions given above fulfil the full allocation condition. Thus, additionally taking the correction of the risk measure for a discrete loss distribution into account, we are able to define IS estimators for the additive tail risk contributions as follows:

$$\widehat{VaR}_q(L_i \mid PL) = \frac{\sum_{k=1}^{s} w_i \cdot LGD_i \cdot D_i^k \, \mathbb{1}_{\{\widehat{VaR}_q(PL)\}}\big(PL^k\big)\, l\big(PL^k\big)}{\sum_{k=1}^{s} \mathbb{1}_{\{\widehat{VaR}_q(PL)\}}\big(PL^k\big)\, l\big(PL^k\big)} \tag{4.5}$$

and

$$\widehat{ES}_q(L_i \mid PL) = \frac{\sum_{k=1}^{s} w_i \cdot LGD_i \cdot D_i^k \, \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)}{\sum_{k=1}^{s} \mathbb{1}_{[\widehat{VaR}_q(PL),1]}\big(PL^k\big)\, l\big(PL^k\big)} + \frac{1}{1-q}\widehat{VaR}_q(L_i \mid PL)\big[\hat{F}_{PL}\big(\widehat{VaR}_q(PL)\big) - q\big]. \tag{4.6}$$

Applying the IS technique outlined above, instead of a plain MC simulation, can lead to substantial variance reduction when estimating VaR, ES and ES contributions. Note, nevertheless, that the problem concerning an efficient estimation of VaR contributions persists, since individual losses conditional on $PL = VaR_q(PL)$ are still rare.

5. Measuring and allocating systemic risk using an analytical approximation

Although the number of simulation runs can be reduced considerably by using IS, the need for an approximative analytical solution has been accentuated repeatedly in the related literature. For the special case of a single-risk factor model and an asymptotically infinitely fine-grained portfolio, there exists an analytical solution for portfolio VaR/ES as well as for the VaR/ES contributions, (see Gordy, 2003). In this asymptotical setting, the idiosyncratic risk is diversified away and the risk contributions are portfolio-invariant. In order to mitigate the underestimation of VaR in finite portfolios, closed-form expressions for a granularity adjustment have been derived by Wilde (2001) and Martin and Wilde (2002). Based on their results, Emmer and Tasche (2003) have determined contributions to the adjusted approximate portfolio VaR. These contributions are portfolio-dependent due to the existence of an undiversified idiosyncratic risk.

The adjustment methodology for VaR and ES has been extended more recently by Pykhtin (2004), who presented an analytical method for an approximative calculation of portfolio VaR and ES in the case of a *multi*-factor Merton framework. Based on his results we derive closed formulae for Euler contributions as partial derivatives of the approximated VaR and ES. Pykhtin's approach is outlined in AppendixB for completeness. The basic idea of his approximative solution is to redefine the multi-factor model presented in section 3 in terms of a comparable one-factor model whose implied portfolio loss distribution is similar to the original one. For this purpose a new "effective" systematic factor $\bar{Y}$ is introduced:

$$\bar{Y} = \sum_{k=1}^{m} \beta_k Z_k, \qquad \sum_{k=1}^{m} \beta_k^2 = 1. \tag{5.1}$$

The tail risk measures can then be approximated by a formula containing Gordy's approximation for a limiting portfolio PL^∞ within the one-factor framework (superscript $\bar{Y}$) augmented by the adjustment term which corrects for the systematic and idiosyncratic risks within the multi-factor setting (denoted with Δ):

$$VaR_q(PL) \approx VaR_q^{approx}(PL) = VaR_q^{\bar{Y}}(PL^\infty) + \Delta VaR_q(PL)$$

and

$$ES_q(PL) \approx ES_q^{approx}(PL) = ES_q^{\bar{Y}}(PL^\infty) + \Delta ES_q(PL).$$

The components of these formulae are given in AppendixB, equations (B.6, B.7) and (B.14, B.15) respectively.

The analytical approximations above can be used to derive the additive contributions associated with the portfolio tail risk measures. Under the assumptions of an infinitely fine-grained portfolio and only one systematic risk factor, the contribution of an institution i to the VaR of the limiting portfolio, as defined by equation (B.6), would be completely portfolio-invariant because of the following result:

$$\frac{\partial}{\partial w_i} VaR_q^{\bar{Y}}(PL^\infty) = LGD_i \cdot p_i(y_q). \tag{5.2}$$

The y_q denotes a realisation of $\bar{Y}$ associated with the $(1-q)$ quantile of its Gaussian probability distribution: $y_q = \Phi^{-1}(1-q)$ and $p_i(y_q)$ is the probability of distress conditional on $\bar{Y} = y_q$:

$$p_i(y_q) = \Phi\left(\frac{\Phi^{-1}(p_i) - b_i y_q}{\sqrt{1-b_i^2}}\right). \tag{5.3}$$

In addition to the stand-alone marginal risk contribution, a portfolio-dependent contribution arises according to equation (B.7) by reason of the following multi-factor granularity adjustment:

$$
\begin{aligned}
\frac{\partial}{\partial w_i}\Delta VaR_q(PL) &= \left\{2\left[\big(PL^\infty(y_q)\big)'\right]^2\right\}^{-1} \\
&\times \Bigg\{-\frac{\partial}{\partial w_i}\big(\mathrm{var}(PL \mid \bar{Y}=y_q)\big)'\big(PL^\infty(y_q)\big)' + \big(\mathrm{var}(PL \mid \bar{Y}=y_q)\big)'\frac{\partial}{\partial w_i}\big(PL^\infty(y_q)\big)' \\
&\quad + \left[\frac{\partial}{\partial w_i}\big(\mathrm{var}(PL \mid \bar{Y}=y_q)\big)\big(PL^\infty(y_q)\big)' - \mathrm{var}(PL \mid \bar{Y}=y_q)\frac{\partial}{\partial w_i}\big(PL^\infty(y_q)\big)'\right] \\
&\quad \times \left(\frac{\big(PL^\infty(y_q)\big)''}{\big(PL^\infty(y_q)\big)'} + y_q\right) + \mathrm{var}(PL \mid \bar{Y}=y_q)\big(PL^\infty(y_q)\big)'\frac{\partial}{\partial w_i}\left(\frac{\big(PL^\infty(y_q)\big)''}{\big(PL^\infty(y_q)\big)'}\right)\Bigg\}.
\end{aligned}
\tag{5.4}
$$

The derivatives on the right-hand side of equation (5.4) are given by in AppendixC.

In order to calculate an approximation of the marginal VaR contribution of the ith bank as a percentage of its own exposure, we just need to add up (5.2) and (5.4):

$$
\frac{\partial}{\partial w_i}VaR_q(PL) \approx \frac{\partial}{\partial w_i}VaR_q^{approx}(PL) = \frac{\partial}{\partial w_i}VaR_q^{\bar{Y}}(PL^\infty) + \frac{\partial}{\partial w_i}\Delta VaR_q(PL). \tag{5.5}
$$

The approximative VaR contributions defined as

$$
VaR_q^{approx}(w_i \mid PL) = w_i \frac{\partial}{\partial w_i}VaR_q^{approx}(PL), \tag{5.6}
$$

satisfy the full allocation property:

$$
VaR_q^{approx}(PL) = \sum_{i=1}^{n} VaR_q^{approx}(w_i \mid PL).
$$

Similar to the previous results, risk contributions based on $ES_q^{\bar{Y}}(PL^\infty)$ in (B.14) would be portfolio-invariant:

$$
\frac{\partial}{\partial w_i}ES_q^{\bar{Y}}(PL^\infty) = \frac{LGD_i}{1-q}C^{Gauss}\big(p_i, 1-q; b_i\big). \tag{5.7}
$$

An additional adjustment term corrects for the systematic risk within the multi-factor setting and for the undiversified idiosyncratic risk. This adjustment term can be obtained by the partial differentiation of equation (B.15) with respect to the exposure weights:

$$\frac{\partial}{\partial w_i}\Delta ES_q(PL) = -\frac{\phi(y_q)}{(1-q)}\left[2\big(PL^{\infty}(y_q)\big)'\right]^{-1}$$
$$\times\left[\frac{\partial}{\partial w_i}\big(\mathrm{var}(PL \mid \bar{Y}=y_q)\big)\big(PL^{\infty}(y_q)\big)' - \mathrm{var}(PL \mid \bar{Y}=y_q)\frac{\partial}{\partial w_i}\big(PL^{\infty}(y_q)\big)'\right]. \tag{5.8}$$

So we can approximate the ES contribution as a percentage of institution i's exposure

$$\frac{\partial}{\partial w_i}ES_q(PL) \approx \frac{\partial}{\partial w_i}ES_q^{approx}(PL) = \frac{\partial}{\partial w_i}ES_q^{\bar{Y}}(PL^{\infty}) + \frac{\partial}{\partial w_i}\Delta ES_q(PL) \tag{5.9}$$

or alternatively as a percentage of the total portfolio exposure

$$ES_q^{approx}(w_i \mid PL) = w_i\frac{\partial}{\partial w_i}ES_q^{approx}(PL). \tag{5.10}$$

Again, the approximative ES contributions in (5.10) satisfy the full allocation property:

$$ES_q^{approx}(PL) = \sum_{i=1}^{n} ES_q^{approx}(w_i \mid PL).$$

6. The Performance of the IS method versus Pykhtin's approximation

In this section we compare the performance of the IS method versus Pykhtin's approximation. This analysis is based on empirical data in order to make it more realistic and and to increase its validity. The empirical data are described in subsection 6.1 and will be used again in later sections. Subsection 6.2 contains the main results.

6.1. Empirical expected default rates and other model inputs

The dataset used for the empirical analysis comprises a sample of the world's largest banks over a time span from January 1997 to January 2010. The number of banks varies between 54 and 86 depending on IPOs, mergers and data availability. The one-year probability of default is estimated on a monthly basis by the expected default frequency (EDF) from Moody's KMV CreditEdge. The EDFs range from 0.01% to 19% with the median value 0.07% before September 2008 and 0.32% afterwards. We set the *EAD* equal to the book value of the bank's liabilities, also obtained from CreditEdge on a yearly basis. We transform the yearly observations into monthly data by linear interpolation. Missing a reliable estimate of a bank's *LGD*, we use the value of 100% for all banks[1] which implies the maximum loss rate. Since the *LGD* is modelled as a deterministic variable, the risk contributions are linear in *LGD* and, therefore, its specific number does not affect our main results.

We define the systematic risk factors by the geographical region in which the bank is headquartered. Table 1 presents summary statistics of the size distribution of banks in the

[1] Tarashev et al. (2010) set the *LGD*-rate to 55% without giving any reasons.

sample across 6 regions (Europe, North America, South America, Africa, Japan, Asia and Pacific excluding Japan). The banks listed in the table account for about 2/3 of the worldwide banking industry assets in 2007/2008 (approximated by assets of the largest 1,000 banks as reported by IFSL (2010)).

We have set the asset return correlation within the groups to the asset return correlation average of 42%, estimated for large banks on the basis of the Moody's KMV GCorr module, as reported by Tarashev et al. (2010, p 21). It implies homogenous factor loadings $a_i = \sqrt{0.42}$ $\forall i$. The heterogeneity in the dependence structure arises from the correlation between the region-specific systematic risk factors. The off-diagonal elements of their correlation matrix have been estimated from monthly returns of the Dow Jones Total Market (DJTM) total return indices for the banking sector in the respective geographical regions, obtained from Datastream. The estimates are reported in Table 2. They reveal substantial differences in the correlation between geographical regions which support our choice of a multi-factor instead of a single-factor model.

6.2. Performance results for the ES estimation

We compare how the proposed simulation and analytical techniques perform with regard to the calculation of the portfolio tail risk and marginal risk contributions by a three-step approach. Firstly, we run 100 plain MC and IS simulation scenarios, each scenario comprising $10,000$ independent replications of the portfolio loss variable PL. This enables us to compare the accuracy of the simulation methods and to compute pointwise empirical confidence intervals for the quantities under consideration. Secondly, we approximate the tail risk and risk contributions analytically based on the results in section 5. Thirdly, we check whether these approximated values fall into the confidence intervals, obtained in the simulation.

Figure 1 exemplifies the considerable gain in precision compared to a plain MC simulation when estimating the loss distribution and the portfolio tail risk by means of IS. Figure 2 compares the performance of the MC and IS estimators for the ES contributions. The box-and-whiskers plots clearly show a substantial reduction in variability using IS.[2]

The analytical method performs reasonably well for the calculation of portfolio tail risk with a tendency to underestimate. For 59% of observations in our case study the approximated value lies within the 90% error-interval of the IS simulation, as shown exemplary in Figure 3 for the 20 latest observations. Additionally, Figure 4 illustrates the analytically approximated ES in comparison with the IS-based estimate along with the relative difference between them. The Pykhtin's formula exhibits the poorest performance in the period of a relatively low system-wide risk. The best performance is on the contrary at the peak of the crises. A more detailed performance test for the multi-factor-adjustment technique was carried out by Pykhtin (2004). Among other things, Pykhtin shows that the accuracy of the approximation improves as the risk factor correlation increases and as the relative weight of the largest exposure in the portfolio decreases.

[2] We refer to Glasserman (2006) for further numerical examples on the performance of the IS algorithm concerning the problem of estimating the tail risk contributions for credit portfolios.

Table 1: Liability (LBS) size distribution of all banks within the sample at the beginning of 2008, aggregated by country.

Region	Country	Number of banks	Aggregate LBS	
			billion USD	% of total
EU	Austria	1	265	0.49
	Belgium	2	1,286	2.39
	Denmark	1	606	1.12
	France	3	5,571	10.33
	Germany	4	4,155	7.71
	Greece	1	111	0.21
	Iceland	1	64	0.12
	Italy	3	2,146	3.98
	Netherland	2	3,179	5.90
	Norway	1	244	0.45
	Russia	1	146	0.27
	Spain	3	1,988	3.69
	Sweden	3	1,122	2.08
	Switzerland	2	3,079	5.71
	United Kingdom	6	8,758	16.24
AMN	Canada	5	2,093	3.88
	USA	11	7,274	13.49
AMS	Brazil	3	352	0.65
AFR	South Africa	3	322	0.60
JP	Japan	5	4,577	8.49
AS&P	Australia	5	1,589	2.95
	China	10	3,456	6.41
	Hong Kong	2	212	0.39
	India	2	305	0.57
	Singapore	3	353	0.65
	South Korea	3	654	1.21
Total		86	53,907	100

Table 2: The matrix of estimated Pearson's correlation coefficients for the composite factors, $\rho_{Y_{reg(i)},Y_{reg(j)}}$. All p-values are less then 1%

	EU	AMN	AMS	AFR	JP	AS
EU	1.00	0.80	0.65	0.63	0.44	0.85
AMN		1.00	0.42	0.44	0.39	0.73
AMS			1.00	0.50	0.46	0.68
AFR				1.00	0.32	0.62
JP					1.00	0.45
AS						1.00

Figure 1: Log-lin graph of the portfolio loss tail function in September 2008 estimated via Monte Carlo simulation and importance sampling. In each case, the three curves show the mean and a pointwise 95% confidence interval computed on the basis of 100 independent scenarios.

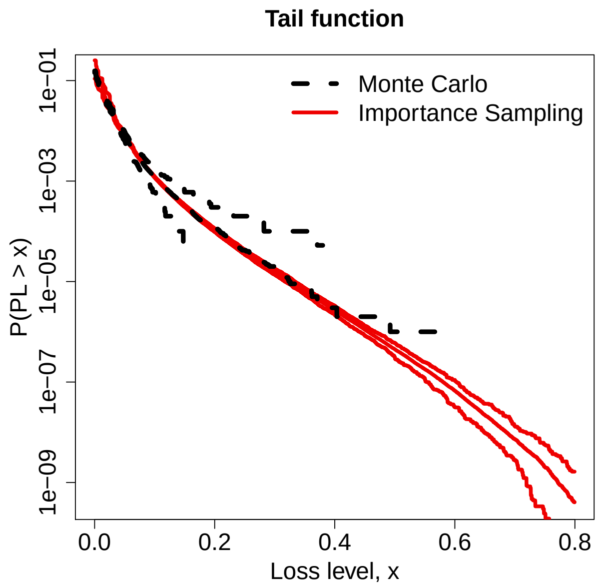


Figure 2: Comparison of the summary statistics for expected shortfall (ES) contributions in September 2008, estimated via Monte Carlo (MC) and importance sampling (IS). The box-and-whiskers plots are based on 100 independent scenarios. The five largest contributions are given as a fraction of the total liabilities of 84 banks in the portfolio.

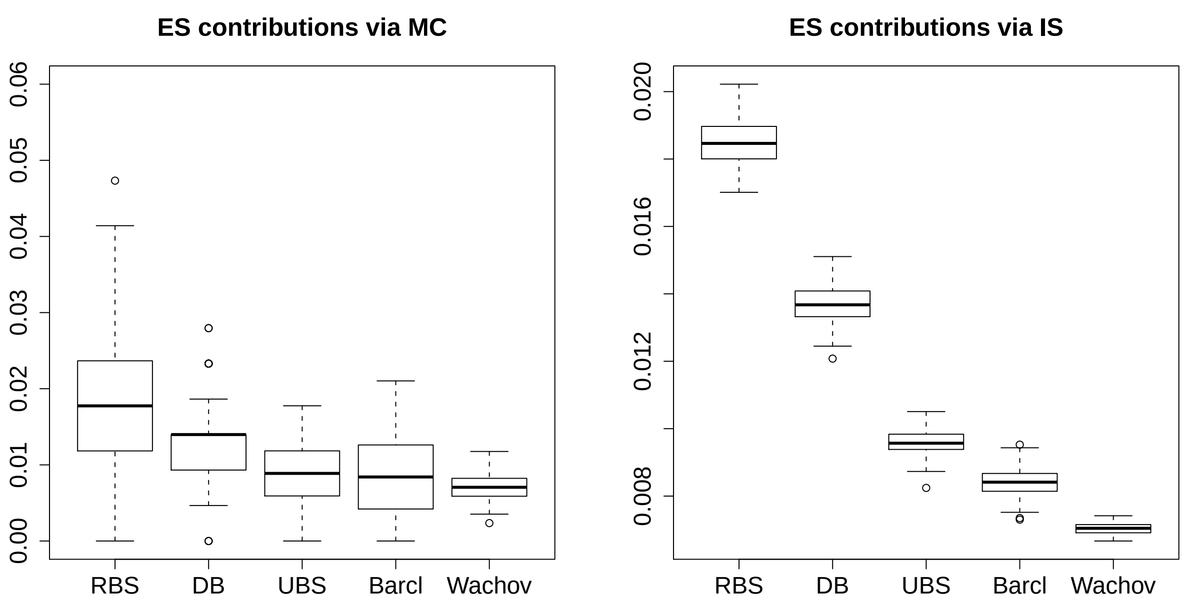


Figure 3: Comparison of expected shortfall (ES) values estimated via importance sampling with those approximated analytically. The date is always given in the lower right-hand corner. ES is given as a percentage of the total portfolio liabilities. For each month the analytically approximated portfolio ES is indicated by a triangle pointing down to its numerical value, whereas the patterns enclose 90% of all 100 sampled ES values with the mean indicated by a circle. Only the 20 latest observations are shown.

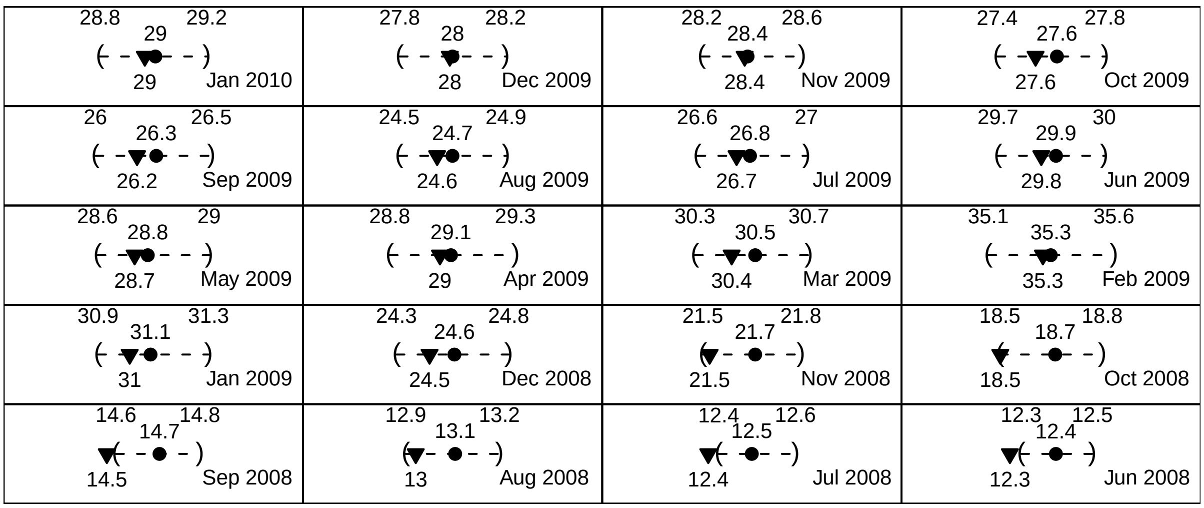


Figure 4: Comparison of expected shortfall (ES) values estimated via importance sampling with those approximated analytically. ES is given as a percentage of the total portfolio liabilities. Also plotted is the relative difference between the two estimates.

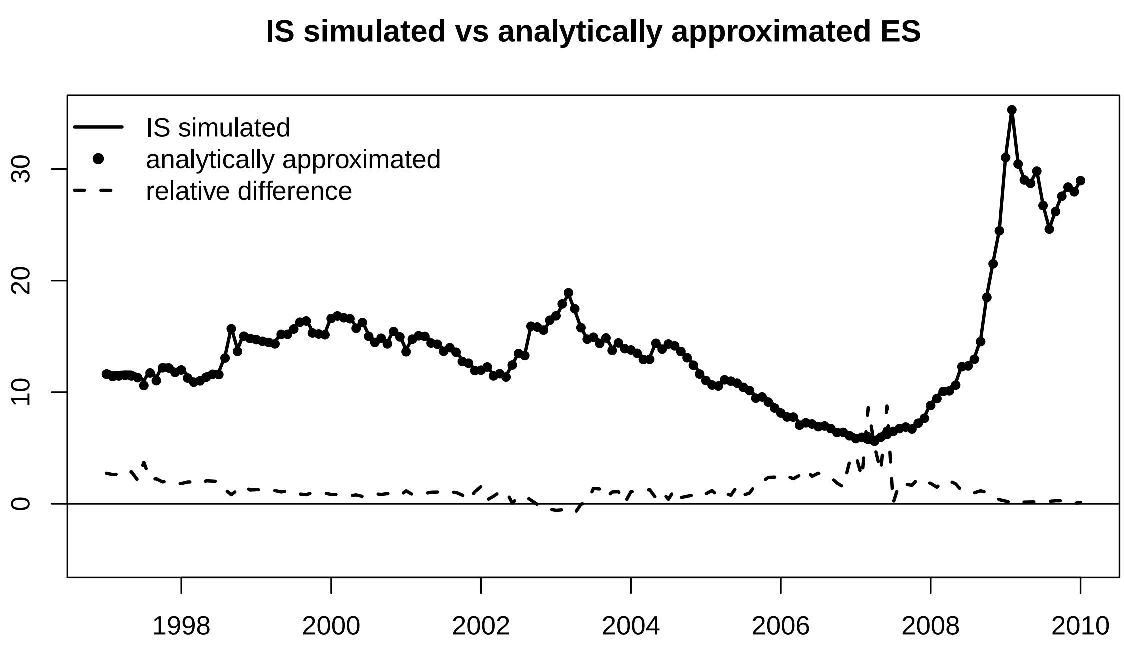
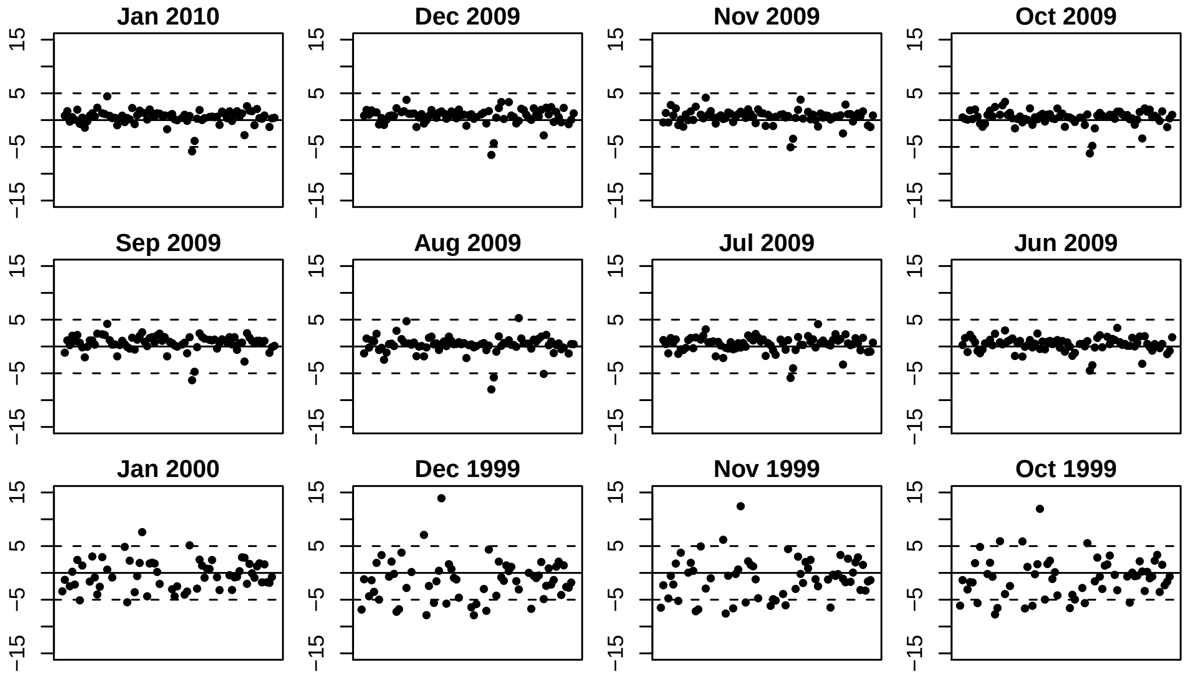


Figure 5: The relative difference (in percent) between the contributions to the expected shortfall estimated via importance sampling and those approximated analytically. The banks are on the x-axis.



Regarding the approximation of the individual risk contributions derived in section 5, we report in Figure 5 results on the relative difference between the IS-estimated and analytically approximated ES contributions for some of those time periods when the relative difference at the portfolio level was less than 1%. The relative difference for most of the contributions presented is smaller than 5%.

Whereas the analytical approximation of the portfolio tail measures performs well, the results of the analytical approximation of individual risk contributions should be interpreted with caution. While the precision of the IS estimator can be improved by simply performing more simulation runs, the analytical results depend on the portfolio granularity and the correlation structure. Although the contributions calculated using marginal method are generally guaranteed to be positive for positively correlated risk, it may not longer be the case within the Pykhtin's modified setting. Due to the changes in the correlation structure in the course of the model transformation from the multi-factor (3.1) to the one-factor (5.1) setting, the impact of the largest exposures in the portfolio may be overvalued. This effect would then be compensated by reduced and possibly even negative contributions of small-sized exposures in order to satisfy the additivity property. Thus, the analytical approximation for the risk contributions can, for instance, be used to get preliminary, approximative results when accuracy is not the issue but only the computational burden.

7. Drivers of systemic risk and systemic importance

The impact of the risk drivers within a credit portfolio framework has been analysed in great detail in the literature on credit risk. In context of systemic risk, Tarashev et al. (2010)

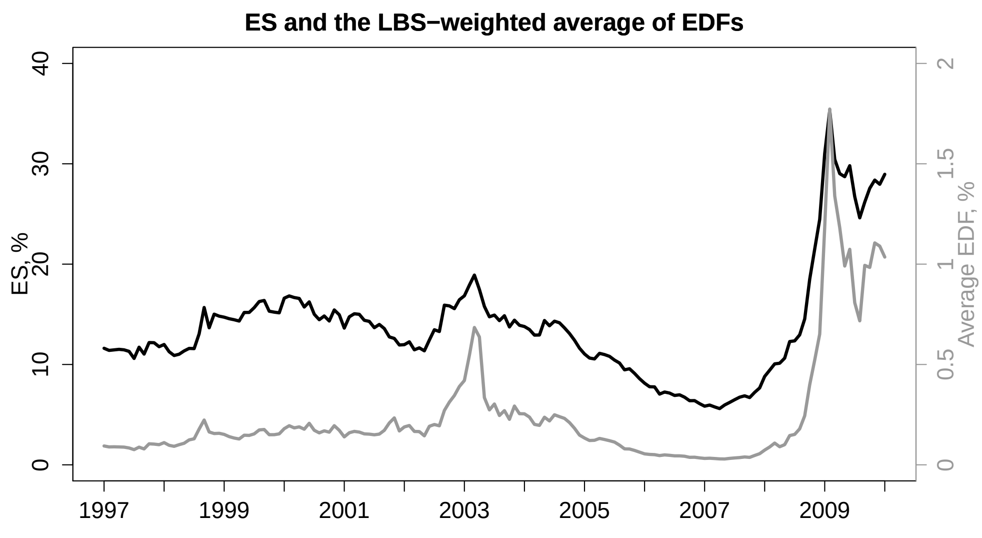
presented some stylised examples for hypothetical financial systems in order to examine the sensitivity properties of the system-wide ES. Thereby the authors applied the one-factor Merton/Vasicek framework with common factor loadings. The key messages from their work were:

- The level of systemic risk increases with the individual probabilities of default.
- Greater bank concentration of the financial system, caused either by the increasing disparity of the relative size of banks or by their decreasing number, raises systemic risk.
- Higher sensitivity to the common factors (captured by the asset correlation) increases the likelihood of joint failures and raises the tail risk.

In this section we further explore the impact of the individual probabilities of default, the relative size of institutions and the asset correlation on the portfolio tail risk measure.

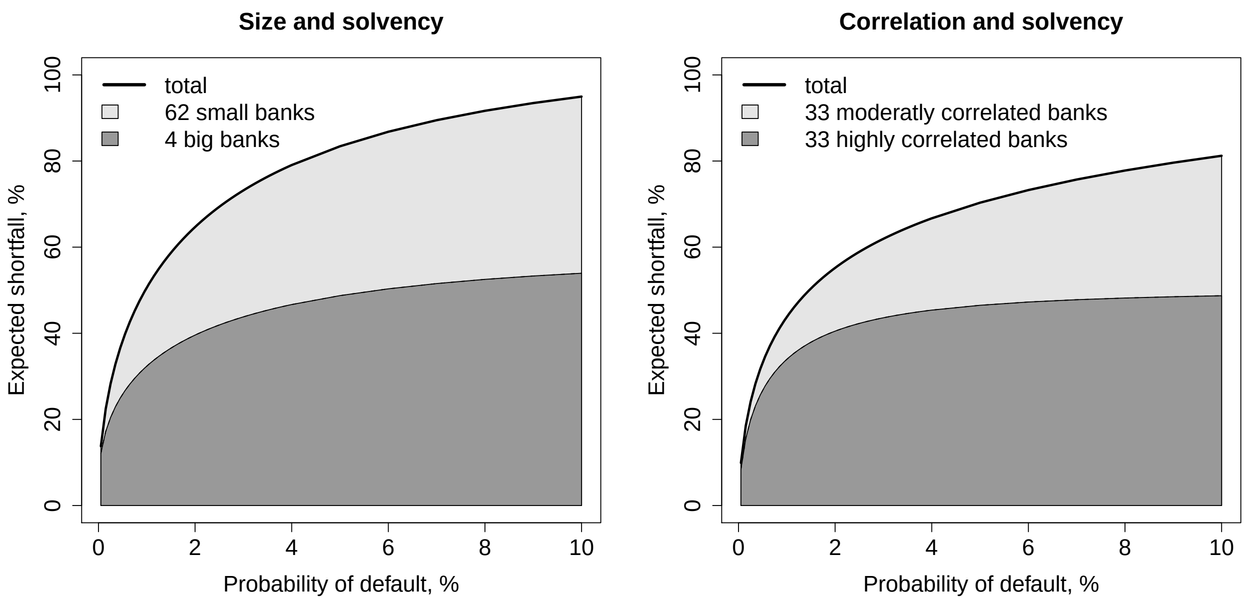
We can confirm the first finding of Tarashev et al. (2010) by using the empirical dataset and model inputs from section 6.1. Figure 6 shows the ES over the sample period and the weighted average of the underlying EDF figures. The ES matches very closely the pattern of the average estimated probabilities of default. Only if we assume a constant EDF, the ES would follow the pattern of banks' liabilities, which had been more or less steadily increasing until mid-2008.

Figure 6: Evolution of the portfolio expected shortfall (ES, black lines, left axes) expressed as a percentage of the total portfolio liabilities (LBS). Also plotted is the weighted average of EDFs (gray lines, right axes); the weights are the shares of individual banks in the total LBS.



Regarding the bank concentration of the financial system, we can isolate the impact of the relative size for different levels of default probabilities by a simulation exercise based on a stylised portfolio. For this purpose we consider the special case of a single-factor model and define a stylised banking system populated by 66 banks which all share the same probability

Figure 7: Systemic importance of two groups of banks with different size (left-hand plot) and different exposure to the single systematic factor (right-hand plot). Each of the two groups accounts for half of the total portfolio exposure.



of default. All the banks can be separated into two groups, each accounting for 50% of the overall liabilities. We define one group of 62 equally-sized small banks and another group of 4 equally-sized big banks. To keep the exposures to the single systemic factor constant across the system, we set the pairwise asset correlation to 42%. The results for this financial system are presented in the left-hand panel of Figure 7. Notwithstanding the fact that both groups are equally sized, the group of big banks accounts for more than 50% of the overall ES according to its greater bank concentration. This effect is even more distinctive for small probabilities of default (below 1%) which are typical for the banking sector. Hence, among relatively sound institutions the banks with larger exposures at distress affect the overall tail risk disproportionately more heavily. Rising probabilities of default *ceteris paribus* lead to a higher overall tail risk and have a positive impact on the systemic importance.

For a heterogenous empirical portfolio, like the one introduced in section 6.1, it is more difficult to distinguish between the impact of different risk drivers. Therefore, we estimate the cross-sectional Spearman's correlation[3] between an institution's contribution to the expected portfolio loss (which is just the product of the institution's relative size and its default probability) and its share in the portfolio ES. The correlation coefficients rage between 81% and 96% with the median of 91%. Thereby the impact of the size is more pronounced (the medium correlation between the size and the ES contribution is about 90%) than the impact

[3]Note, that the Spearman's correlation increases in magnitude as the two variables become closer to being perfect monotone (possibly non-linear) functions of each other.

of a bank's default risk (the medium correlation in this case is about 26%). These results are in line with those for the stylised portfolio.

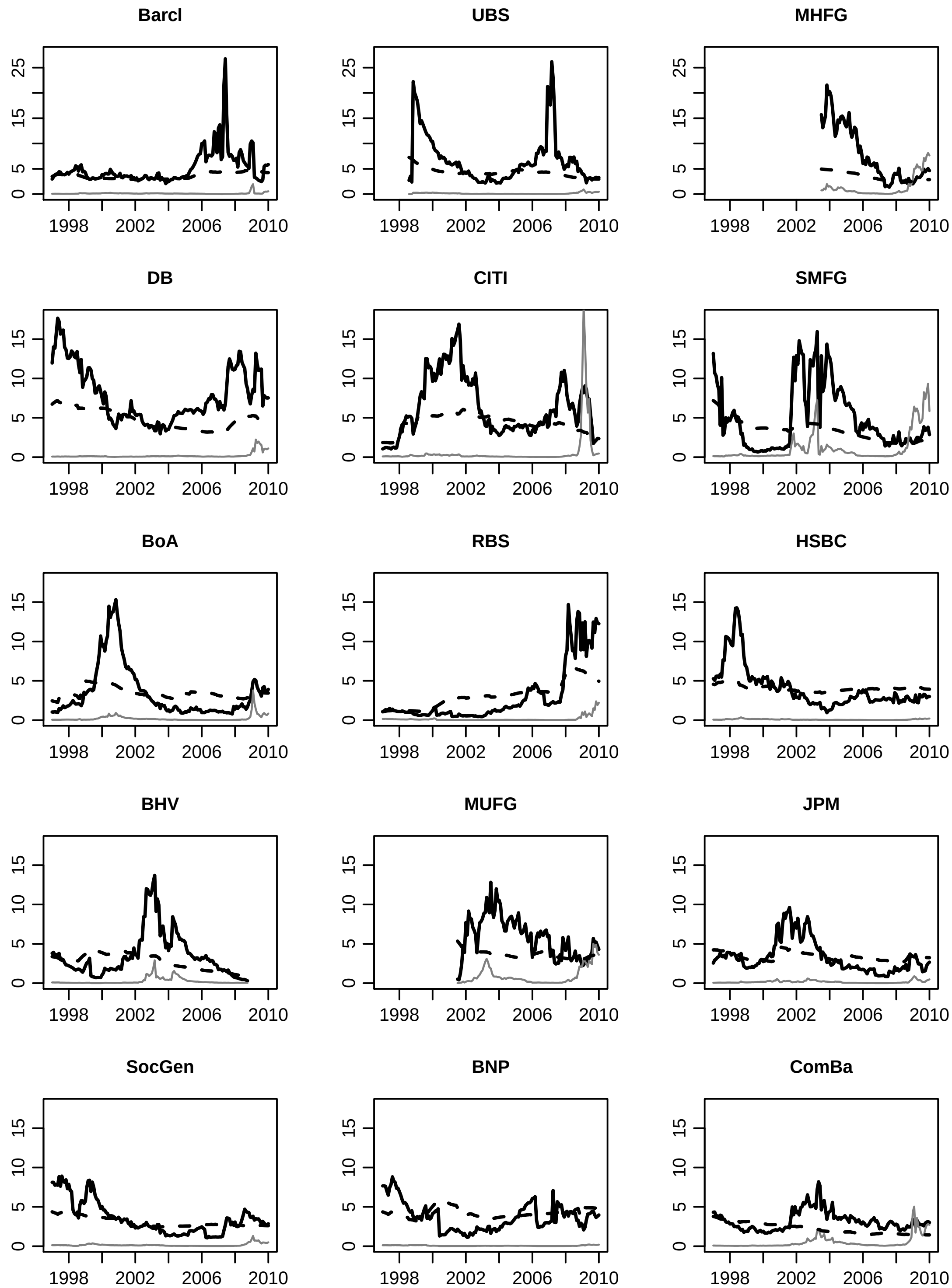
We use the stylised portfolio again in order to explore the impact of the sensitivity to the common factors which refers to the third finding of Tarashev et al. (2010). For this purpose we isolate the impact of the asset correlation as a risk driver of the systemic risk and banks' systemic importance. We divide the portfolio into two homogeneous groups comprising 33 equally-sized banks each. The first group is only moderately exposed to the systematic factor with the pairwise within-group asset correlation of 20%. The banks assigned to the second group are instead highly correlated with a coefficient of 60%. The right-hand panel of Figure 7 illustrates the intuitive result that a higher sensitivity to the systemic factor, i.e., a higher asset correlation, is linked to a higher systemic importance, since the probability of joint failures increases. This leads to a higher level of tail risk. Again, it is worth noting that the tail risk contribution of the group of banks with a high sensitivity to the systemic risk factor increases faster within the range of small default probabilities than is the case for the other group.

Turning back to the empirical dataset from section 6.1, we investigate additionally the sample path of the banks' *relative* ES contributions in comparison with the banks' relative size and their EDFs. These variables are plotted in Figure 8 for the 15 banks with the historically largest risk contributions within the sample period. The comparison between size and systemic risk contribution of those major banks along the time axis shows that their relative ES contribution often considerably exceeds their share in the total liabilities, indicating an overproportional contribution to the risk of the whole system. Because the level of the system-wide tail risk is closely related to the overall default risk in the system, as has already been shown in Figure 6, the changes in the levels of banks' contributions to the tail risk are linked to the changes in the individual default probabilities. The corresponding correlation estimates confirm that statement: Apart from one bank with a significant negative coefficient and 6 banks with insignificant coefficients (at the 95% level), Spearman's correlation along the time axis ranges from a minimum of 18% to a maximum of 96% with the median observation of 65%.

Summarising, our findings point to the following interpretation of the risk drivers' impact. Firstly, changes over time in the joint probabilities of default affect changes in the overall level of the systemic risk much more than changes in the size distribution of the portfolio. Therefore, the financial soundness of the institutions under consideration and the correlations between them seem to be the main drivers of the systemic risk. Secondly, given a particular level of tail risk at a particular point in time, the distribution of the risk contributions depends strongly on the size distribution among the banks.

Overall, the link between the size of a financial institution and its systemic risk contribution is not that obvious due to an interplay of risk drivers in both dimensions: cross-sectional and over time. Despite a pronounced positive relation between the size of an institution and its contribution to systemic risk in the cross-sectional context, size alone cannot be considered as a reliable proxy of a bank's systemic importance. When the size of a bank increases, its systemic importance can increase or decrease depending on changes in its own and other institutions' risk drivers. In the static context likewise, not only a bank's individual characteristics affect its systemic importance, but also the composition of the system which the bank is a part of. This can be seen, for instance, from equation (B.10), which is a part of the

Figure 8: Dynamics of the banks' individual shares in the portfolio expected shortfall (solid black lines) in comparison with the EDFs (solid gray lines) and individual shares in the total portfolio liabilities (dashed lines).



approximative solution for an ES contribution (5.9): A bank's contribution depends on the size of other banks in the system as well as on the respective default probabilities and asset correlations with another banks. These findings confirm the need to study systemic risk in a portfolio context instead of on a single entity basis. Tailoring macro-prudential instruments simply to the size of a financial institution would be at best an incomplete assessment of its systemic risk. It would miss key aspects of the risk that it can pose to the real economy and society.

8. Policy tools – A capital charge for systemic risk and a mitigant of procyclical effects

A macro-prudential regulation should address both dimensions of the systemic risk, as is underlined by Borio (2009) among others:

The *cross-sectional dimension*, addressed in subsection 8.1, relates to the distribution of the aggregate risk in a financial system at a given point in time. The corresponding policy issue consists in the calibration of prudential instruments according to the level of the overall risk in the system and according to the contributions of individual institutions to the system-wide tail risk.

The *time dimension*, addressed in subsection 8.2, covers the evolution of the aggregate risk over time. The corresponding policy issue is to find a way to reduce the possible procyclicality of regulatory tools based on a measure of the system-wide financial risk.

8.1. Cross-sectional implementation

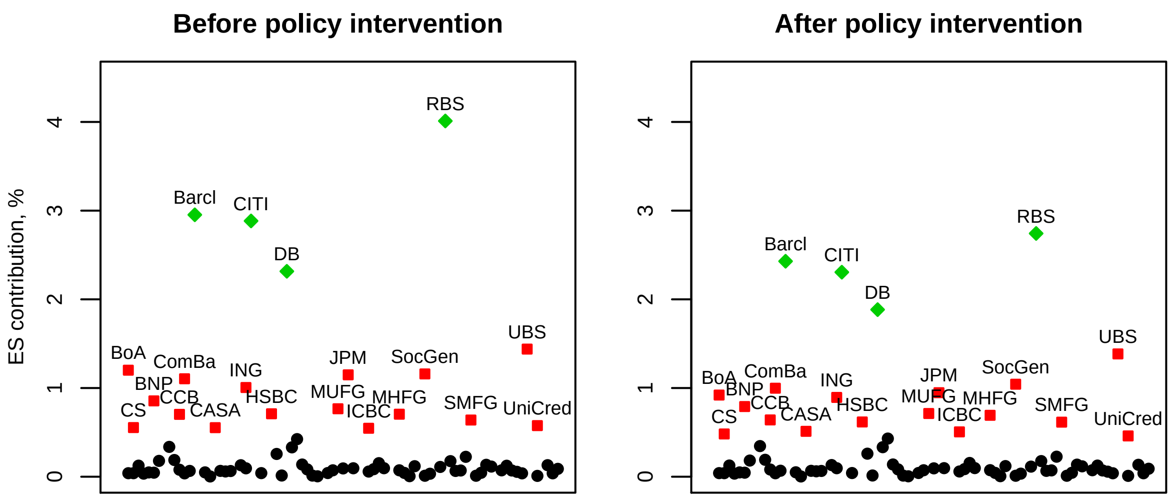
For implementing a systemic risk charge in the cross-section, a key challenge is how to internalize the negative externalities caused by financial institutions. This goal is achieved by using the institutions' contributions to the systemic risk as the building block. In this section we put forward a stylised example illustrating how a capital surcharge for systemic-risk can help the regulator to reduce the tail risk amount.

We consider the situation arising in January 2009 as an example. At this point of time, the portfolio comprises of 80 banks from the sample in section 6.1. The ES of the portfolio amounts to 31.38% of the system-wide liabilities or \$17,447bn. Individual ES contributions vary between 0.03‰ and 4% or \$1.8bn and \$2,230bn.

As the measurement and allocation of systemic risk involve model uncertainty and estimation errors, it may be advisable not to require a bank-specific surcharge on a continuous scale. Instead, a less granular approach may be preferable: For instance dividing the institutions into three different categories A, B and C according to *systemic risk ratings*.[4] We apply a simple k-means clustering procedure on the ES contributions in order to define those categories. The k-means method aims to partition the dataset into k groups. The grouping is done by minimising the sum of squares of distances between the data points and the clusters' centroids. The results are illustrated in the left-hand panel of Figure 9.

[4]The IMF (2010, Chapter 2) presents an approach under which regulators assign systemic risk ratings to each institution based on the amount of system-wide capital impairment that a hypothetical default of each institution would bring to bear on the financial system. Institutions with a higher systemic risk rating would be assessed as having higher capital surcharges. The level of capital surcharges would be predetermined – perhaps having to be agreed upon in international forums.

Figure 9: The landscape of systemically important banks before and after the policy intervention. 3 groups of banks have been identified by the k-means clustering method according to their contributions to the portfolio ES in January 2009. The banks are on the x-axis.



Let us categorise the four banks belonging to the first group and indicated by diamonds as A-rated, "highly systemically important" institutions. This group holds 20% of the total assets of the system and contributes 38.8% to the overall ES. The individual ES contribution of every bank in this group equals or exceeds 2% of the portfolio exposure. The squares mark the second category comprises 16 B-rated, "moderately systemically important" banks. They individually contribute between 0.5% and 1.5% to the portfolio exposure. This second group of banks holds 47% of total assets and shares 43.5% of the overall tail risk. The remaining 60 banks are indicated by solid circles. They share 34% of total assets and 17.7% of the portfolio ES. Those banks account for risk contributions of less than 0.5% each and will be denoted as C-rated, "systemically less relevant" institutions.

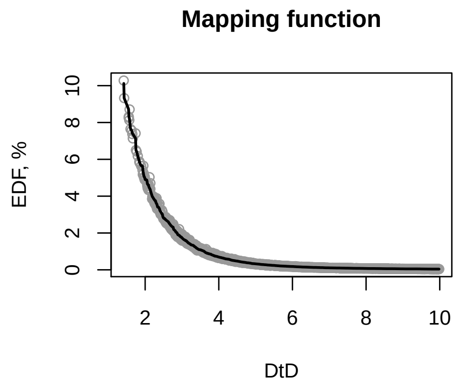
We assume that the capital held by banks equals the amount of capital required by the regulator. Therefore, any capital charge for systemic risk will require an increase of capital and cannot be drawn from an existing "free" capital buffer on top of the regulatory minimum requirements. Furthermore, we assume that the systemic risk charge does not affect neither the size of a bank's balance sheet nor its exposure to the systematic factors (or asset correlations). The new capital requirements only affect the debt-to-equity ratio as the banks substitute their (short-term) debt by capital. In this case, the rising capital charge would leave the asset value of banks unchanged according to the Modigliani-Miller capital structure irrelevance principle.

Within Merton's framework (Merton, 1974), the following functional link between default probability and leverage ratio applies

$$p_i = \Phi\left(\frac{\ln(DPT_i/AVL_i) - \mu_{AVL_i}}{\sigma_{AVL_i}}\right),$$

where DPT denotes the default point, AVL – the market value of assets, μ_{AVL} – the expected

Figure 10: Mapping the distance to default into the EDF for a one-year time horizon.



asset return and σ_{AVL} – the volatility of asset value. DPT_i is defined in such a way, that a drop in the market value of bank i's assets below DPT_i triggers the default of the bank. Moody's KMV model, which builds on Merton's framework, calibrates DPT as a weighted average of long-term and short-term liabilities. The model operates with the so called distance to default (DtD) :

$$DtD_i = -\frac{\ln(DPT_i/AVL_i)}{\sigma_{AVL_i}}. \tag{8.1}$$

Using the CreditEdge data on EDFs and DtD, we can approximate the mapping function between DtD and EDF as shown in Figure 10.

We further assume that for each rating category, national regulators have agreed upon a certain level of capital surcharges. In our simplified example, additional capital requirements are set to 50% of the current microprudential capital requirements for "highly systemically important" institutions, to 25% for "moderately systemically important" institutions and to nil for "systemically less relevant" banks.

According to the assumptions we made, the policy intervention results in a modified capital structure of the systemically important banks, reducing their short-term debt, as well as the default point, exactly by the amount of the additionally raised capital. By inserting the new DPT into (8.1) we find the corresponding distance to default and map it into the EDF. We use the new set of EDFs to run the analytical approximation of the portfolio ES and risk contributions after the policy intervention.

The new input parameters change the overall view of the systemic risk landscape, as demonstrated in the right-hand panel of Figure 9. While the total capital in the system rises by 17% and the liabilities decrease by 0.75%, the system-wide ES undergoes a reduction of 13.93%. In nominal terms, the amount of capital is increased by $418bn with the effect that the ES of the financial system is reduced by $2,431bn. In other words the systemic risk charge reduces the system-wide risk by a factor of about six. The marginal, USD-denominated tail risk contributions of the banks with ratings A or B decrease. The total risk contribution of the A-rated banks declines by 30.91% from $6,762bn to $5,165bn and the contribution of the B-rated banks by 12.71% from $7,598bn to $6,741bn. In response to the changes in the portfolio structure, the total USD-denominated risk contribution of the 60 "systemically less relevant" banks increases slightly by 0.70% from \$3,088bn to \$3,109bn.

The empirical example relies on a relatively coarse differentiation between three groups of banks depending on their systemic risk contribution. The capital surcharge has been set arbitrarily since this example does not offer a methodology to determine a continuous, bank-specific *systemic capital charge (SCC)*. In the remaining of this section we present an approach that translates a bank's contribution to the ES of the financial system into a firm-specific capital charge.

We consider the i-th institution in the year t that is subject to minimum capital requirements (MCR). The key idea is to charge the difference between a "pure" systemic risk contribution and the original regulatory minimum capital requirement. If the micro-prudential regulatory capital requirement exceeds the systemic risk contribution of a bank, then no add-on for systemic risk is charged. The following equation summarises this definition of an ES-based SCC:

$$SCC_i(PL,t) = \max\left\{EAD_i(t)\frac{\partial}{\partial w_i(t)}ES_q(PL,t) - MCR_i(t), 0\right\}. \tag{8.2}$$

According to the figures on the total regulatory capital holding by the banks, for which we could obtain the corresponding data from Bankscope, in 2006/2007 86% out of 63 banks were well capitalised in the sense that they reported capital exceeding $MRS_i(t) + SCC_i(PL,t)$ as defined in (8.2). In 2008/2009 the same was only true for 15% out of 72 banks.

Additional capital requirements as in (8.2) are generally in line with the FSB's recommendations to strengthen the loss absorbency of systemically important banks (see FSB, 2010). However, as pointed out by Gauthier et al. (2010), computing macro-prudential capital requirements is more complex than computing risk contributions itself. The simple formula (8.2) suggests setting the capital surcharges according to the currently observed capital levels and does not take into account the subsequent changes in the overall systemic risk landscape. Once new capital requirements are implemented, the banks' probabilities of default (and potentially also the asset correlations) decline resulting in lower tail risk and changed absolute and relative risk contributions. For this reason Gauthier et al. (2010) suggest an iterative procedure to solve for the fixed point at which the capital allocation in the system is consistent with the banks' risk contributions. Such reallocation of the capital not only means that the undercapitalised banks raise capital or de-leverage, but also that the overcapitalised banks increase their leverage. A superior approach not simply based on the *re*allocation of the given total capital, would require the knowledge of the optimal total level of capital required in the banking system to withstand a predefined shock. The optimal amount of capital is not necessarily to cover systemic risk completely, since the tail risk in the system can be far too high to be fully backed with capital.[5] Therefore, the level of the total capital requirements could be on average lower than the amount of the ES. This means that a certain fraction of the systemic risk will still be borne by the public.

[5]During the time period under consideration the system-wide exposure (i.e., $\sum_{i=1}^{n} LBS_i(t)$) increased from 23 to 100 percent of the global GDP whereas the amount of the tail risk was varying between 6.8 and 29 percent of the global GDP according to $ES_{q=0.999}$ or between 3.7 and 16 percent according to $ES_{q(t)}$, which we will introduce in the following. The IMF's figures on the world GDP were taken.

An alternative to a capital charge for system-wide risk can be regular payments by the systemically important banks into a bank stabilisation fund. Although, this policy measure does not promote strengthening of the banks' capital basis, it has the advantage that the money paid into the fund would be available in a crisis situation without the need to tap the taxpayer's purse, e.g., for financing certain bridge banks. A yearly amount to be paid into the fund could be attributed to individual banks by employing the banks' relative contributions to the tail risk of the whole banking sector. Further refinements could be contemplated. For example, in order to relieve the strain on savings banks and other mostly deposit-taking institutions, exposures could be reduced by the amount of ensured deposits, which could be achieved by setting $LGD_i < 1$ accordingly.

Both, a capital charge for system-wide risk and a stabilisation fee would reduce the competitive advantages to become systemically important. The latter statement provides an incentive for the systemically important institutions to reduce their share in the system-wide tail risk, which is a desirable effect.

8.2. Smoothing the path of the tail risk measure over time

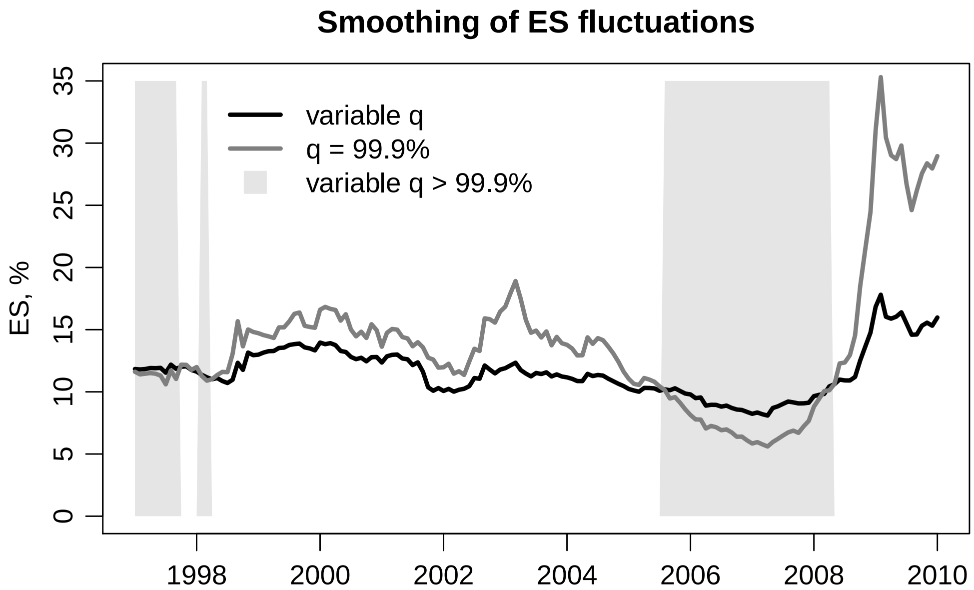
Within the presented framework the evolution of systemic risk over time is mainly driven by the co-movement of the probabilities of default in the banking sector. In Figure 6 we have seen how the use of point-in-time estimates of the default probability based on market prices can induce procyclicality in the tail risk measure. Market-based measures suggest that the system is strongest in times when market volatility is below average and market participants accumulate large amounts of risk. During an economic downturn or turbulent markets, probabilities of default (and asset correlations) increase and the tail risk measure increases. The described effect by itself is not a problem when considering the portfolio expected shortfall as a systemic risk indicator for the banking sector. In this regard the utilisation of more forward-looking estimates of default probabilities would be rather an advantage. However, in order to establish such macro-prudential tools as systemic capital surcharges a procyclical pattern of the underlying risk measure may deem undesirable.

To take the procyclical effect into account, we suggest to use a time-varying level of the regulator's tail-risk tolerance $q(t)$ for calculating expected shortfall, denoted by $ES_{q(t)}$. We suggest to link $q(t)$ to the cross-sectional exposure-weighted average of default probabilities in the banking sector:

$$q(t) = 1 - \sum_{i=1}^{n} \frac{EAD_i(t)}{\sum_{j=1}^{n} EAD_j(t)} p_i(t).$$

The confidence level for the portfolio under consideration ranges from 98.23% to 99.97% with a median of 99.85%. It exceeds 99.9% during a boom in a financial sector and declines below this level otherwise. Therefore, it leans against the cycle and leads to the mitigation of possible procyclical effects of regulatory tools based on the expected shortfall risk measure. For example, the capital surcharges based on $ES_{q(t)}$ would be higher during the "good" times than the surcharges based on $ES_{q=0.999}$ and vice versa. This effect is indicated with shaded areas in Figure 11. This figure illustrates a considerable reduction in the variability of the portfolio-level ES, which we achieve by means of the time-varying confidence level. The range of variation shrinks from 5.61% – 35.31% to 8.09% – 17.81% of total liabilities. It is also worth noting, that using joint default probabilities instead of individual ones and allowing

Figure 11: Evolution of the portfolio ES calculated according to the time-varying confidence levels $q(t)$ (black line) versus ES at the constant confidence level $q = 99.9\%$ (gray line).



for varying (default) correlations would amplify the observed effect even further. Thereby, employing more forward-looking estimates of default probabilities instead of the EDFs would help to rise or loosen capital requirements early enough in anticipation of an upcoming boom or bust.

9. Summary and conclusions

Addressing the system-wide risk of a financial system by macro-prudential regulation requires an approach that internalizes the potential costs of a systemic failure. We develop such an approach by assessing the systemic risk of the financial system and by allocating this risk to individual banks while the emphasis is on the allocation of systemic risk to individual banks. We employ for this purpose the Euler allocation principle that is widely used in the risk management of financial institutions.

In this paper a financial system is modeled as a portfolio consisting of those banks in the global financial system which may be deemed systemically relevant. From a public purse perspective we model systemic risk in terms of the expected shortfall (ES) of this portfolio. The expected losses conditional on exceeding a given level of regulatory tolerance reflect the potential costs posed to society in the low-probability event such as a systemic crisis, when the institutions may draw on the (explicit or implicit) guarantees given on their debt.

The portfolio approach used has the additional advantage that the modelling requirements are based on standard risk management techniques and the basic data requirements are similar to those under the internal ratings based (IRB) approach of Basel II. For the major financial institutions, the method provides an assessment tool of systemic importance, based on publicly available information including market prices. Moreover, the model can be applied to smaller, not publicly traded institutions as well, provided that their probabilities of default and their exposures to common risk factors can be estimated based on available information.

After the tail risk of the whole financial system has been quantified by means of the system-wide ES, it is allocated to individual banks based on their marginal risk contributions. An important advantage of this method is the full allocation property, which means that the sum of systemic risk contributions attributed to individual institutions equal the system-wide risk in the aggregate. For the purpose of simulation of the portfolio loss function, upon which the calculation of the portfolio ES and the risk contributions is based, we adopt a two-stage importance sampling method. The main advantage of this variance reduction technique over the plain Monte Carlo method is a considerable gain in efficiency when simulating such rare events as large portfolio losses. We also derive an analytical solution for a fast approximation of risk contributions based on a formula for the tail risk of a limiting portfolio with a multi-factor granularity adjustment.

Having conducted an empirical study based on a sample of large international banks, we find that in the cross-sectional dimension the systemic importance of a financial institution is indeed tightly linked to the institution's relative size. But since the formal linkage is non-linear and portfolio-dependent, size alone should not be considered as a reliable proxy of systemic importance. Other risk drivers, such as institutions' probabilities of default and their exposures to common risk factors, have to be taken into consideration when assessing systemic importance within a portfolio framework.

As to the assessment of financial firms' systemic importance, we can abstain from the binary approach, whereby some firms would be considered of systemic importance and others would not, which would leave room for regulatory arbitrage. By means of individual tail risk contributions, the binary concept can be refined to a desirable degree either by introducing several systemic rating categories or by the utilisation of a direct functional link between an institution's marginal contribution to the systemic risk and its degree of systemic importance.

Relying on the marginal ES contributions as a measure of the institutions' systemic importance, policy tools can be adjusted accordingly. A possible capital-related policy option would be to impose a systemic capital charge as the amount of the systemic risk contribution not covered by the minimum capital requirements. Increasing overall risk-based capital requirements would reduce the probability of systemically important banks becoming distressed. An alterative non-capital based policy option involves charging a stabilisation fee that flows into a systemic risk fund. This would cover the externalities in a systemic crisis and dampen the incentives of financial institutions to become more systemically important. Thereby a total yearly amount that has to be paid into the fund can be defined at the system level in a counter-cyclical manner. It will then be allocated among the institutions according to their shares in the system-wide ES.

Regarding the time dimension of the systemic risk, we have successfully implemented a time-varying confidence level of the ES risk measure in order to smooth the evolution of the ES over time. This approach can help to mitigate possible procyclical effects of regulatory tools based on this measure of systemic risk.

Summarising, the portfolio approach, which we put forward for modelling a system of financial institutions, can help to understand the complex nature of systemic risk regarding its cross-sectional dimension as well as its evolution over time. Further theoretical and empirical research, however, is required to ensure that systemic-risk related policy means are viable and robust before they are put into practice.

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AppendixA. Importance sampling algorithm for the portfolio loss distribution

In this appendix we briefly describe the adopted two-stage IS algorithm for simulation of the portfolio loss distribution and refer to Glasserman and Li (2005) for further details. If we have a realisation of the systematic factors, than its natural to increase the likelihood of conditional defaults and, therefore, of tail portfolio losses. So the first step should be to shift the conditional loss distribution into the region $[x_q, 1]$ by increasing conditional default probabilities.

To make the conditional expected loss equal the threshold x_q we set the conditional default probabilities $p_i(Y_i)$ (see (5.3)) to their exponentially tilted values $p_i(Y_i, \theta)$, which depend on the tilting parameter θ:

$$p_i(Y_i; \theta) = \frac{e^{\theta \cdot w_i \cdot LGD_i} p_i(Y_i)}{1 - p_i(Y_i) + e^{w_i \cdot LGD_i \cdot \theta} p_i(Y_i)}. \tag{A.1}$$

The optimal $\theta \equiv \theta_{x_q}(\mathbf{y})$ can be found by solving:

$$\theta_{x_q}(\mathbf{y}) = \left\{ \theta \ : \ \sum\nolimits_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(y_i; \theta) = x_q \right\}. \tag{A.2}$$

If $x_q > E[PL \mid \mathbf{y}]$, then $\theta_{x_q}(\mathbf{y})$ is positive and the tilted default probabilities $p_i(y_i; \theta_{x_q}(\mathbf{y}))$ are greater than the original ones, leading to larger portfolio losses. Otherwise, $\theta_{x_q}(\mathbf{y})$ is negative and should be set to zero in order to estimate the tail risk, because there is no advantage in reducing $p_i(y_i)$. So the appropriate choice of the tilting parameter in our setting is:

$$\theta^{+}_{x_q}(\mathbf{y}) = \max\{0, \theta_{x_q}(\mathbf{y})\}. \tag{A.3}$$

The default events in a portfolio are typically positively correlated. They tend to occur simultaneously, driven by systematic factors. So a second step is essential to further reduce the variance of IS estimates: the multivariate Gaussian distribution of the systematic factors $\mathbf{Y} = (Y_1 \dots Y_n)'$ should be transformed such that "bad" realisations (negative values, in our case) occur more frequently leading to more defaults in the portfolio.

The transformation could be accomplished by shifting the mean of $\mathbf{Y}$ from $\mathbf{0}$ to $\boldsymbol{\mu}$, leaving the initial correlation matrix (denoted by Σ) unchanged. Depending on x_q, the new mean vector can be chosen according to the solution of the following maximisation problem:

$$\boldsymbol{\mu}_{x_q} = \arg\max_{\mathbf{y}} \left\{ -\theta x_q + C_{PL|\mathbf{Y}}(\theta) - \frac{1}{2}\mathbf{y}'\Sigma^{-1}\mathbf{y} \right\}, \tag{A.4}$$

with $C_{PL|\mathbf{Y}}(\theta) = \sum_{i=1}^{n} \ln\left(1 - p_i(y_i) + e^{w_i \cdot LGD_i \cdot \theta} p_i(y_i)\right)$ being the cumulant generating function of the conditional portfolio loss distribution.

It is important to accentuate the fact that there is no need for a repetitive computation of shifting and tilting parameters for numerous different loss levels. Although the parameters μ_{x_q} and $\theta^{+}_{x_q}(\mathbf{y})$ depend on a particular loss quantile, it is sufficient for a practical implementation to choose only one value of x_q. This loss level should be located in the tail, close to $VaR_q(PL)$ and can be chosen on the basis of a short preliminary MC simulation run or the approximative analytical solution we will present later in the paper. The exact position of the loss threshold is not critical. For the chosen value of x_q the problem (A.4) needs to be solved numerically only once before starting the first simulation run. $\theta^+_{x_q}(\mathbf{y})$ has to be determined once for each realisation $\mathbf{y}$.

Taking this information into account, we suggest the following IS simulation algorithm:

- Choose an appropriate loss level x_q.
- Find $\boldsymbol{\mu}_{x_q}$ by solving (A.4).
- For each replication $k = 1, \dots, s$:
  - generate a realisation $\mathbf{y}$ from $N(\boldsymbol{\mu}_{x_q}, \Sigma)$;
  - calculate $p_i(y_i) = \Phi\left(\frac{\Phi^{-1}(p_i)+a_i Y_i}{\sqrt{1-a_i}}\right)$ for $i = 1, \dots, n$;
  - find $\theta^+_{x_q}(\mathbf{y})$ as in (A.3) by solving (A.2);
  - according to (3.3) generate Bernoulli default indicators either by simulating $D_i(y_i) \sim Be\big(p_i(y_i)\big)$ for $i = 1, \dots, n$ directly or by means of X_i in (3.1);
  - calculate portfolio loss PL^k in the kth simulation run as in (3.4);
  - calculate the likelihood ratio $l\big(PL^k\big)$ which equals to

$$\exp\left[-\theta^+_{x_q}(\mathbf{y})PL^k + C_{PL|\mathbf{Y}}\big(\theta^+_{x_q}(\mathbf{y})\big) + \frac{1}{2}\boldsymbol{\mu}'_{x_q}\Sigma^{-1}\boldsymbol{\mu}_{x_q} - \boldsymbol{\mu}'_{x_q}\Sigma^{-1}\mathbf{y}\right].$$

- Calculate the empirical cumulative distribution function for the portfolio loss according to
$$\hat{F}_{PL}(x) = 1 - \frac{1}{s}\sum\nolimits_{k=1}^{s} \mathbb{1}_{(x,1]}\big(PL^k\big)\, l\big(PL^k\big), \quad x \in [0, 1].$$

AppendixB. The model of Pykhtin

The "effective" systematic factor $\bar{Y}$ introduced in (5.1) is the same for all institutions in the portfolio. Therefore, the model (3.1) can be rewritten in the following way

$$X_i = b_i\bar{Y} + \sqrt{1 - b_i^2}\,\varepsilon_i \tag{B.1}$$

where $\{\varepsilon_i\}_{i=1,\dots,n}$ are independent standard normal variables, $b_i \equiv a_i \sum_{k=1}^m \alpha_{ik}\beta_k$ are the new factor loadings, and $\sum_{k=1}^m \alpha_{ik}\beta_k$ represents the correlation between Y_i and $\bar{Y}$.

The optimal choice of the coefficients $\{\beta_k\}$ is not obvious. Pykhtin suggested maximising the correlation between Y_i and $\bar{Y}$:

$$\max_{\{\beta_k\}}\left\{\sum_{i=1}^{n} c_i \sum_{k=1}^{m} \alpha_{ik}\beta_k\right\} \qquad \text{w.r.t. } \sum_{k=1}^{m}\beta_k^2 = 1, \tag{B.2}$$

$$c_i = w_i \cdot LGD_i \cdot \Phi\left(\frac{\Phi^{-1}(p_i) + a_i\Phi^{-1}(q)}{\sqrt{1 - a_i^2}}\right). \tag{B.3}$$

Thereby, differentiating the Lagrange function

$$L\big(\{\beta_k\}, \lambda\big) = \sum_{i=1}^{n} c_i \sum_{k=1}^{m} \alpha_{ik}\beta_k - \lambda\left(\sum_{k=1}^{m} \beta_k^2 - 1\right)$$

and putting the partial derivatives to zero yields

$$\beta_k = \frac{1}{2\lambda}\sum_{i=1}^{n} c_i\alpha_{ik}, \quad k = 1, \ldots, m,$$

$$\lambda = \frac{1}{2}\sqrt{\sum_{k=1}^{m}\left(\sum_{i=1}^{n} c_i\alpha_{ik}\right)^2} = \frac{1}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \rho_{Y_i,Y_j}}.$$

In doing so, we can eliminate $\{\alpha_{ik}\}$ from the equation:

$$b_i = \frac{a_i}{2\lambda}\sum_{k=1}^{m}\alpha_{ik}\sum_{j=1}^{n} c_j\alpha_{jk} = \frac{a_i}{2\lambda}\sum_{j=1}^{n} c_j\rho_{Y_i,Y_j}. \tag{B.4}$$

The factor loadings $\{b_i\}$ are all we need to know about the model representation (B.1) in order to carry on with the calculation of VaR and ES.

Representation (B.1) has just the form of a one-factor model. In the case of the limiting portfolio, provided that $\sum_{i=1}^{n} w_i^2 \to 0$ while $n \to \infty$, the portfolio loss rate in a one-factor model is a function of the systematic risk factor

$$PL^{\infty}(\bar{Y}) = E[PL \mid \bar{Y}] = E\left[\sum_{i=1}^{n} w_i \cdot LGD_i \cdot D_i \mid \bar{Y}\right] = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(\bar{Y}), \tag{B.5}$$

and the corresponding asymptotic solution for the portfolio VaR is well known from Gordy (2003):

$$VaR_q^{\bar{Y}}(PL^{\infty}) = PL^{\infty}(y_q) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(y_q), \tag{B.6}$$

where $p_i(y_q)$ is the conditional probability of distress given in (5.3).

The expression above describes the conditional mean of the portfolio loss distribution depending on a "bad" realisation of $\bar{Y}$. The second order Taylor approximation of $VaR_q(PL)$[6] require an additional correction for the conditional variance. Therefore, we augment expression (B.6) with an adjustment term, which corrects for the portfolio granularity in the

[6] See proposition 2.2 in Emmer and Tasche (2003).

multi-factor setting and can be written as:

$$\Delta VaR_q(PL) = \tag{B.7}$$
$$-\frac{1}{2\big(PL^\infty(y_q)\big)'}\left[\big(\mathrm{var}(PL \mid \bar{Y} = y_q)\big)' - \mathrm{var}(PL \mid \bar{Y} = y_q)\left(\frac{\big(PL^\infty(y_q)\big)''}{\big(PL^\infty(y_q)\big)'} + y_q\right)\right].$$

The derivatives of the limiting portfolio used in (B.7) can be found in AppendixC, expressions (C.1) to (C.4).

So far there has been nothing special concerning the representation of the one-factor model (B.1) in terms of a convex combination of $\{Z_k\}$, as given by (5.1). However, in order to obtain a formula for $\mathrm{var}(PL \mid \bar{Y} = y_q)$ we need to take into account that asset returns are actually not independent given a realisation of the effective risk factor $\bar{Y}$. It can be seen from the following representation:

$$X_i = b_i \bar{Y} + \sum_{k=1}^{m}(a_i \alpha_{i,k} - b_i \beta_k) Z_k + \sqrt{1 - a_i^2}\,\epsilon_i.$$

In fact, the conditional asset correlation between two distinct institutions i and j is given by

$$\rho_{i,j}^{\bar{Y}} = \frac{\rho_{i,j} - b_i b_j}{\sqrt{1 - b_i^2}\sqrt{1 - b_j^2}}. \tag{B.8}$$

Although meaningless as a correlation coefficient, expression (B.8) has to be extended to cover the case $j = i$, i.e., $\rho_{i,i}^{\bar{Y}} = (r_i^2 - b_i^2)/(1 - b_i^2)$.

Asset returns are only independent conditional on the whole set of systematic factors $\{Z_k\}$. Thus, according to the law of total variance, we may decompose $\mathrm{var}(PL \mid \bar{Y} = y_q)$ to separate the variance of the limiting portfolio loss $\mathrm{var}^\infty(\cdot)$ from the effect of granularity $\mathrm{var}^{GA}(\cdot)$:

$$\begin{aligned}\mathrm{var}(PL \mid \bar{Y} = y_q) &= \mathrm{var}^\infty(PL \mid \bar{Y} = y_q) + \mathrm{var}^{GA}(PL \mid \bar{Y} = y_q)\\ &= \mathrm{var}\big(E[PL \mid \{Z_k\}] \mid \bar{Y} = y_q\big) + E\big[\mathrm{var}(PL \mid \{Z_k\}) \mid \bar{Y} = y_q\big].\end{aligned} \tag{B.9}$$

Thereby, $E[PL \mid \{Z_k\}]$ corresponds to the limiting portfolio loss in the multi-factor setting (see (3.1) and (3.2)) given by:

$$\begin{aligned}PL^\infty(\{Z_k\}) = E[PL \mid \{Z_k\}] &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot p_i(\{Z_k\})\\ &= \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \Phi\left(\frac{\Phi^{-1}(p_i) - a_i \sum_{k=1}^{m} \alpha_{ik} Z_k}{\sqrt{1 - a_i^2}}\right).\end{aligned}$$

Taking into account the conditional correlation parameter specified in equation (B.8), the multi-factor adjustment terms for the limiting case and for the effect of granularity can be given by

$$\begin{aligned}
&\text{var}^{\infty}(PL \mid \bar{Y} = y_q) = \text{var}\Big(E[PL \mid \{Z_k\}] \mid \bar{Y} = y_q\Big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot \text{cov}\Big(p_i(\{Z_k\}), p_j(\{Z_k\}) \mid \bar{Y} = y_q\Big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \Big[C^{Gauss}\Big(p_i(y_q), p_j(y_q); \rho_{i,j}^{\bar{Y}}\Big) - p_i(y_q)p_j(y_q)\Big],
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
&\text{var}^{GA}(PL \mid \bar{Y} = y_q) = E\big[\text{var}(PL \mid \{Z_k\}) \mid \bar{Y} = y_q\big] \\
&= \sum_{i=1}^{n} \; w_i^2 \cdot LGD_i^2 \cdot E\Big[\Big(p_i(\{Z_k\}) - p_i(\{Z_k\})p_i(\{Z_k\})\Big) \mid \bar{Y} = y_q\Big] \\
&= \sum_{i=1}^{n} w_i^2 \cdot LGD_i^2 \Big[p_i(y_q) - C^{Gauss}\Big(p_i(y_q), p_i(y_Q); \rho_{i,i}^{\bar{Y}}\Big)\Big],
\end{aligned} \tag{B.11}$$

respectively. In the equations above, $C^{Gauss}(\cdot, \cdot; \rho)$ denotes the bivariate Gauss copula with the correlation parameter ρ. It assigns the conditional probability of a simultaneous distress of institutions i and j (extended to include the case $j = i$).

Due to the variance decomposition (B.9), the multi-factor adjustment for the portfolio VaR in equation (B.7) can also be represented as a sum of two terms: one correcting the VaR of the limiting portfolio for the systematic effect in the multi-factor setting ($\Delta VaR_q^{\infty}(PL)$), and another, addressing the granularity ($\Delta VaR_q^{GA}(PL)$). Then, the approximation formula turns out to be:

$$VaR_q(PL) \approx VaR_q^{approx}(PL) = VaR_q^{\bar{Y}}(PL^{\infty}) + \Delta VaR_q^{\infty}(PL) + \Delta VaR_q^{GA}(PL). \tag{B.12}$$

As to the expected shortfall, Pykhtin derived an analytical approximation of the ES using the integral representation (3.6) by setting:

$$\begin{aligned}
ES_q(PL) &\approx \frac{1}{1-q}\int_q^1 \big(VaR_t^{\bar{Y}}(PL^{\infty}) + \Delta VaR_t(PL)\big)dt \\
&= ES_q^{\bar{Y}}(PL^{\infty}) + \Delta ES_q(PL).
\end{aligned} \tag{B.13}$$

The first term in equation (B.13) represents ES in the case of the limiting portfolio within the one-factor framework:

$$ES_q^{\bar{Y}}(PL^{\infty}) = \frac{1}{1-q}\sum_{i=1}^{n} w_i \cdot LGD_i \cdot C^{Gauss}\big(p_i, 1-q; b_i\big). \tag{B.14}$$

The second term is the multi-factor adjustment defined as a linear function of the conditional variance:

$$\Delta ES_q(PL) = -\frac{\phi(y_q)}{2(1-q)} \frac{\text{var}(PL \mid \bar{Y} = y_q)}{\big(PL^\infty(y_q)\big)'}. \tag{B.15}$$

Due to the additivity of the variance components (see equations (B.9), (B.10) and (B.11)), $\Delta ES_q(PL)$ can also be represented as a sum of its systematic and idiosyncratic parts. Therefore, the analytical approximation of the ES can finally be written as:

$$ES_q(PL) \approx ES_q^{approx}(PL) = ES_q^{\bar{Y}}(PL^\infty) + \Delta ES_q^\infty(PL) + \Delta ES_q^{GA}(PL). \tag{B.16}$$

In the case of large portfolios the systematic parts of VaR and ES, i.e., $VaR_q^{\bar{Y}}(PL^\infty) + \Delta VaR_q^\infty(PL)$ and $ES_q^{\bar{Y}}(PL^\infty) + \Delta ES_q^\infty(PL)$, provide a reasonable approximation of the portfolio risk while the idiosyncratic parts, i.e., $\Delta VaR_q^{GA}(PL)$ and $\Delta ES_q^{GA}(PL)$, vanish. However, due to the fact that the portfolio under consideration could be relatively small and perhaps dominated by a few large exposures, the granularity adjustment terms could be nearly as large as the systematic part.

AppendixC. Derivatives used in the analytical approximation of tail risk and risk contributions

The first and second derivatives of the limiting portfolio loss with respect to y, initially used in expression (B.7), are as follows:

$$\big(PL^\infty(y)\big)' = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \big(p_i(y)\big)', \tag{C.1}$$

$$\big(PL^\infty(y)\big)'' = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \big(p_i(y)\big)'', \tag{C.2}$$

and the corresponding derivatives of the conditional probability of distress are:

$$p_i'(y) = -\frac{b_i}{\sqrt{1-b_i^2}} \phi\left(\frac{\Phi^{-1}(p_i) - b_i y}{\sqrt{1-b_i^2}}\right), \tag{C.3}$$

$$p_i''(y) = -\frac{b_i^2}{1-b_i^2} \frac{\Phi^{-1}(p_i) - b_i y}{\sqrt{1-b_i^2}} \phi\left(\frac{\Phi^{-1}(p_i) - b_i y}{\sqrt{1-b_i^2}}\right). \tag{C.4}$$

According to the variance representation as a sum of the limiting portfolio loss variance (B.10) and the granularity adjustment term (B.11), the first derivative of the conditional portfolio variance $\text{var}(PL \mid \bar{Y} = y_q)$ can also be separated into two parts as follows:

$$\big(\text{var}^\infty(PL \mid \bar{Y} = y)\big)' = 2\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot LGD_i \cdot LGD_j \cdot p_i'(y)\,[Q_{ji}(y) - p_j(y)] \tag{C.5}$$

and

$$\left(\mathrm{var}^{GA}(PL \mid \bar{Y} = y)\right)' = \sum_{i=1}^{n} \; w_i^2 \cdot LGD_i^2 \cdot p_i'(y)\,[1 - 2Q_{ii}(y)] \tag{C.6}$$

with

$$Q_{ji}(y) = \Phi\left(\frac{\Phi^{-1}\big(p_j(y)\big) - \rho_{i,j}^{\bar{Y}}\Phi^{-1}\big(p_i(y)\big)}{\sqrt{1 - \big(\rho_{i,j}^{\bar{Y}}\big)^2}}\right). \tag{C.7}$$

Then, the derivatives with respect to an individual exposure weight w_i, initially used to derive the multi-factor granularity adjustment for the VaR contributions in (5.4), can be obtained for both variance components:

$$\begin{aligned}&\frac{\partial}{\partial w_i}\big(\mathrm{var}^{\infty}(PL \mid \bar{Y} = y)\big) \\ &= 2 \cdot LGD_i \sum_{j=1}^{n} w_j \cdot LGD_j \Big[C^{Gauss}\Big(p_i(y), p_j(y); \rho_{i,j}^{\bar{Y}}\Big) - p_i(y)p_j(y)\Big],\end{aligned} \tag{C.8}$$

$$\begin{aligned}&\frac{\partial}{\partial w_i}\big(\mathrm{var}^{GA}(PL \mid \bar{Y} = y)\big) \\ &= 2w_i \cdot LGD_i^2\Big[p_i(y) - C^{Gauss}\big(p_i(y), p_j(y); \rho_{ij}^{\bar{Y}}\big)\Big].\end{aligned} \tag{C.9}$$

The corresponding derivatives of (C.5) and (C.6) are:

$$\begin{aligned}&\frac{\partial}{\partial w_i}\big(\mathrm{var}^{\infty}(PL \mid \bar{Y} = y)\big)' \\ &= 2 \cdot LGD_i \sum_{j=1}^{n} w_j \cdot LGD_j \cdot p_j\prime(y)\big[Q_{ij}(y) - p_i(y)\big] \\ &+ 2 \cdot LGD_i \cdot p_i'(y) \sum_{j=1}^{n} w_j \cdot LGD_j\big[Q_{ji}(y) - p_j(y)\big],\end{aligned} \tag{C.10}$$

$$\frac{\partial}{\partial w_i}\big(\mathrm{var}^{GA}(PL \mid \bar{Y} = y)\big)' = 2w_i \cdot LGD_i^2 \cdot p_i'(y)\big(1 - 2Q_{ii}(y)\big). \tag{C.11}$$

Eventually, the last two expressions also used in the analytical approximation of the risk contributions are the following

$$\frac{\partial}{\partial w_i}\big(PL^\infty(y)\big)' = LGD_i \cdot p_i'(y), \tag{C.12}$$

$$\begin{aligned} &\frac{\partial}{\partial w_i}\left(\frac{\big(PL^\infty(y)\big)''}{\big(PL^\infty(y)\big)'}\right) \\ &= \frac{LGD_i \cdot p_i''(y)\sum_{j=1}^n w_j \cdot LGD_j \cdot p_j'(y) - LGD_i \cdot p_i'(y)\sum_{j=1}^n w_j \cdot LGD_j \cdot p_j''(y)}{\sum_{j=1}^n w_j \cdot LGD_j \cdot p_j'(y)}. \end{aligned} \tag{C.13}$$